Name: ______________________________

St George Girls High School

Trial Higher School Certificate Examination

2015

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen.
- Write your student number on each booklet.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- The mark allocated for each question is listed at the side of the question.
- Marks may be deducted for careless or poorly presented work.

Total Marks – 70

Section I – Pages 2 – 5
10 marks

- Attempt Questions 1 – 10.
- Allow about 15 minutes for this section.
- Answer on the sheet provided.

Section II – Pages 6 – 11
60 marks

- Attempt Questions 11 – 14.
- Allow about 1 hour 45 minutes for this section.
- Begin each question in a new booklet.
- Show all necessary working in Questions 11 – 14.

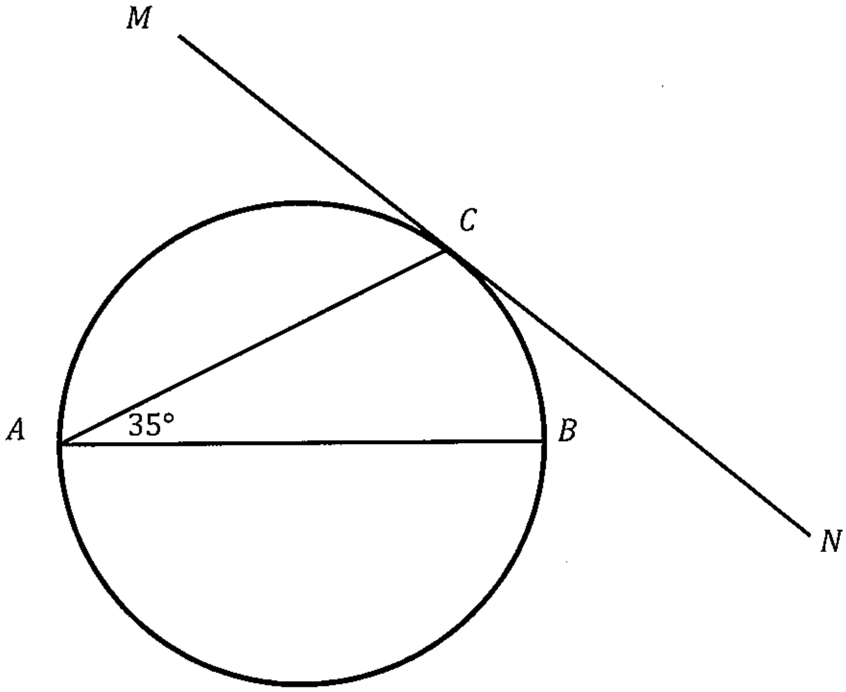
Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

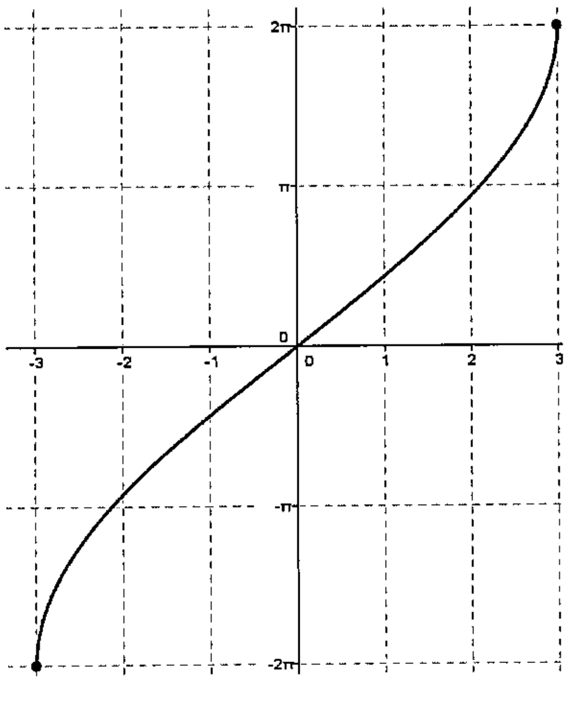
Use the multiple-choice answer sheet for Questions 1–10.

1. In the diagram, AB is a diameter of the circle and MN is tangent to the circle at C. $\angle CAB = 35°$. What is the size of $\angle MCA$?



(A) 35°
(B) 45°
(C) 55°
(D) 65°

2. Which function is graphed below?



(A) $2\pi \sin 3x$

(B) $2\pi \sin^{-1} \frac{1}{3} x$

(C) $4 \sin^{-1} 3x$

(D) $4 \sin^{-1} \frac{1}{3} x$

Section I (cont'd)

3. Find $f^{-1}(x)$, given $f(x) = \frac{3x-3}{x-2}$

(A) $f^{-1}(x) = \frac{3y-3}{x-2}$

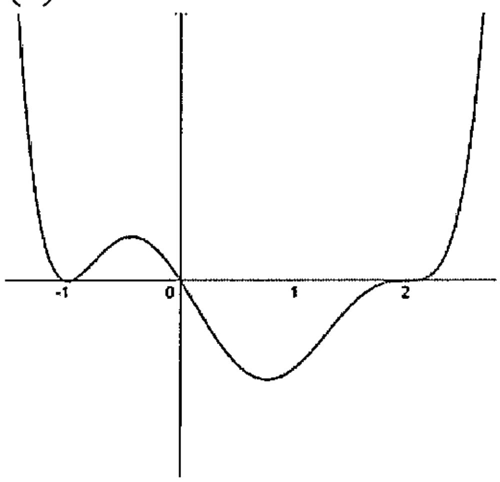
(B) $f^{-1}(x) = \frac{2x-3}{x-3}$

(C) $f^{-1}(x) = \frac{x-2}{3x-3}$

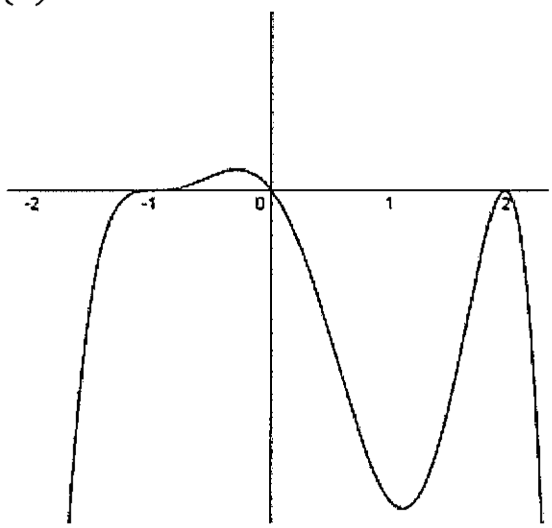
(D) $f^{-1}(x) = \frac{3-3x}{2-x}$

4. Which diagram best represents $y = -x(2-x)^3(x+1)^2$?

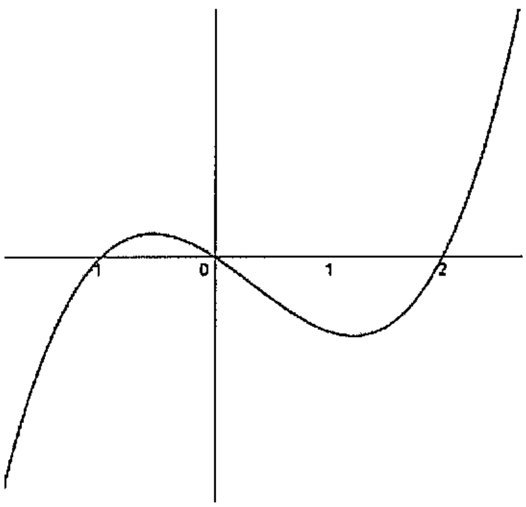
(A)



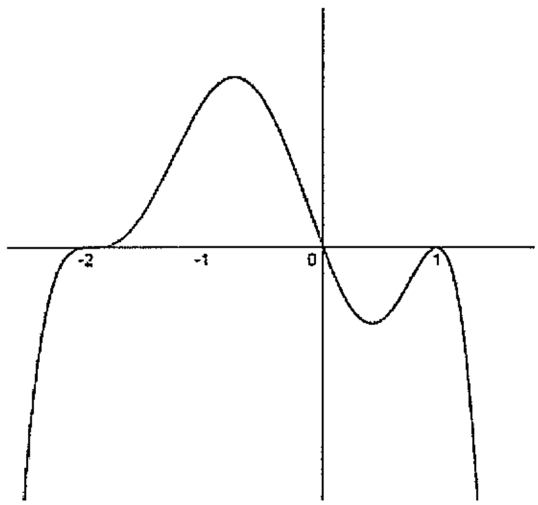
(B)



(C)



(D)



Section I (cont'd)

5. Find k given $x-2$ is a factor of $P(x)=x^3-3x^2+kx+12$

(A) $k=-4$
(B) $k=0$
(C) $k=2$
(D) $k=4$

6. The acute angle between $l_1: 2x-y-3$ and $l_2: y=3x+7$ is closest to:

(A) 15°
(B) 8°
(C) 82°
(D) 45°

7. $\int 2\cos^2 x\, dx$

(A) $\sin x\cos x + x + C$
(B) $-\frac{1}{2}\sin 2x + x + C$
(C) $\frac{2}{3}\cos^3 x + C$
(D) $\frac{-2}{\sqrt{1-x^2}} + C$

8. The velocity of a particle at a position x is $\dot{x} = 2e^{-\frac{x}{2}}$ metres per second.

Calculate the particle's acceleration when its displacement is -2 metres.

(A) $-e$ m/s^2
(B) $-\frac{4}{e^2}$ m/s^2
(C) $-2e^2$ m/s^2
(D) e^2 m/s^2

Section I (cont'd)

9. Find the exact value of $\sin 15°$

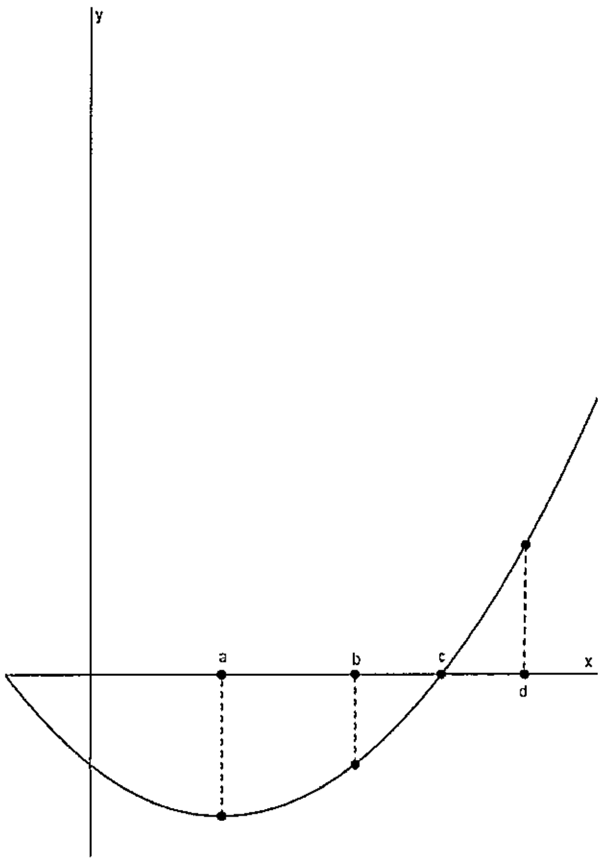
(A) $\dfrac{1}{3\sqrt{2}}$

(B) $\dfrac{2-\sqrt{2}}{2\sqrt{2}}$

(C) $2\left(\sqrt{6}-\sqrt{2}\right)$

(D) $\dfrac{\sqrt{6}-\sqrt{2}}{4}$

10. Given the curve below, Eden intends to use Newtons Method to find an approximation to the root shown. Which initial estimate will not produce a good approximation with this method?



(A) $x_\circ = a$

(B) $x_\circ = b$

(C) $x_\circ = c$

(D) $x_\circ = d$

End of Section I

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hours 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet **Marks**

a) Solve the inequality $\frac{1}{|x-1|} > \frac{1}{2}$ 2

b) Sketch the intersection of $y \geq |x| - 1$ and $y < 1$ 3

c) Given $A(-2, 3)$ and $B(10, 11)$, find the coordinates of the point P which divides the interval AB in the ratio $3:1$. 2

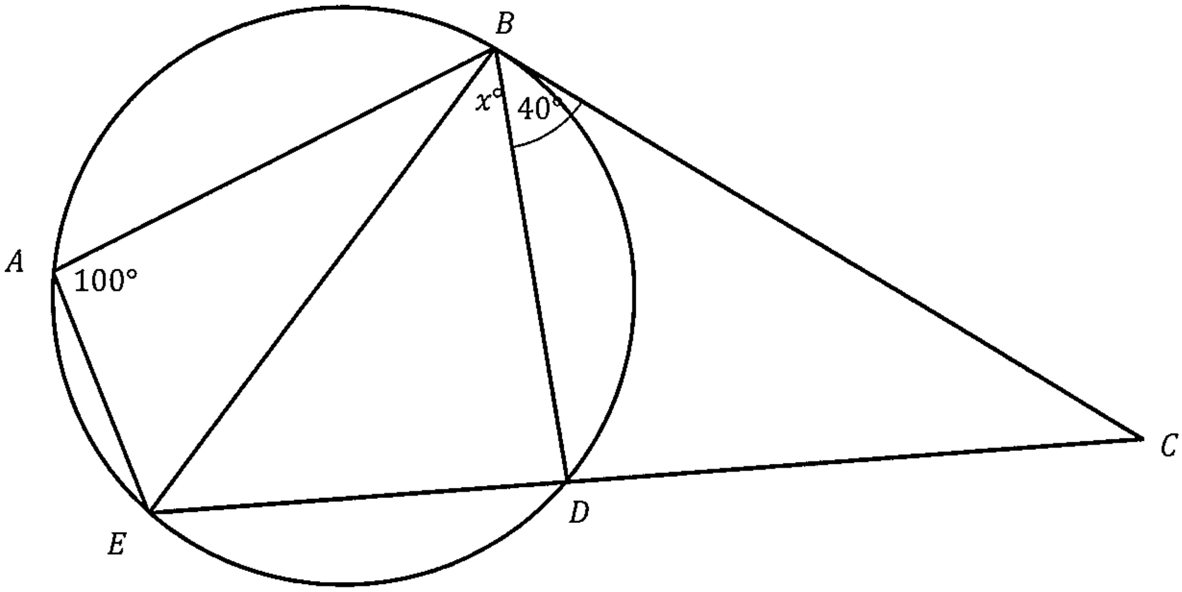
d) You are given 3.6 as an approximate root of the equation $x^3 - 50$. 2
Use one application of Newton's method to find a better approximation.
(to 2 decimal places)

e) If $y = \sin(\ln x)$, find

(i) $\frac{dy}{dx}$ (ii) $\frac{d^2y}{dx^2}$ 1,2

Question 11 (continued) **Marks**

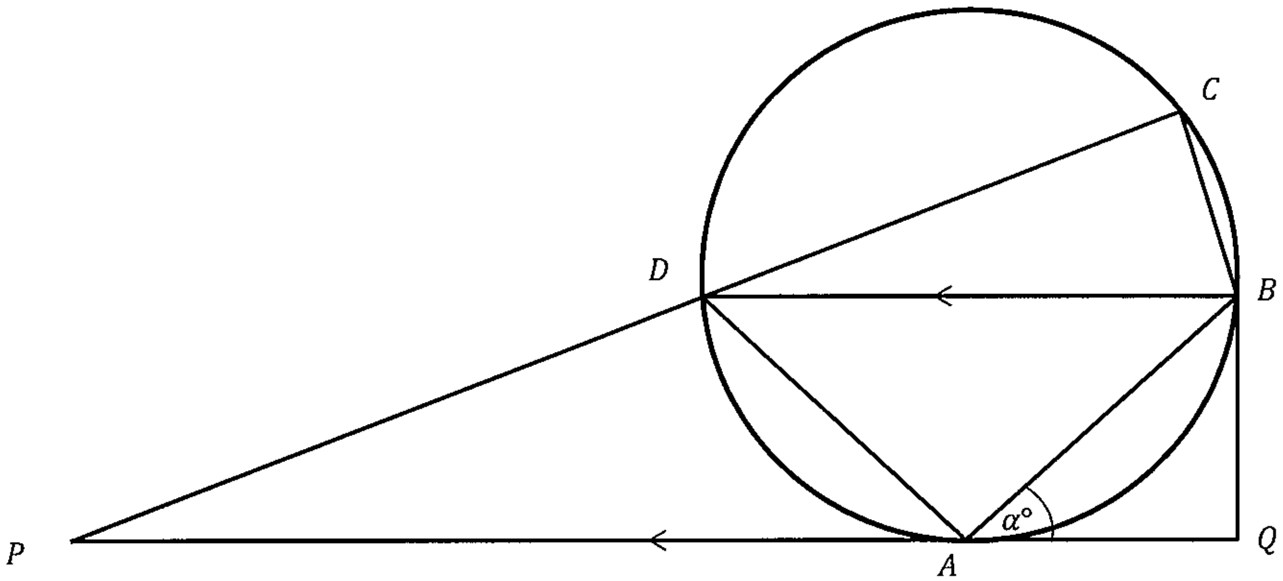
f) BC is tangent to the circle at B. Find the value of x, giving reasons. 3



Question 12 (15 marks) Use a SEPARATE writing booklet **Marks**

a) Solve $3\sin x + 4\cos x = 2.5,\ \ 0 \leq x \leq 2\pi$ 3

b)



The tangents from Q touch the circle at A and B. PC and PQ are straight lines $\angle BAQ = \alpha$.

(i) Copy or trace the diagram into your writing booklet. 1

(ii) Given $PD = 5$ cm and $DC = 7$ cm, calculate the exact length of AP. 1

(iii) Show that $\angle BCD = 2\alpha$. 3

(iv) Show that $PQBC$ is a cyclic quadrilateral. 2

Question 12 (continued) **Marks**

c) The rate at which a body warms in air is proportional to the difference between its temperature T and the constant temperature A of the surrounding air. This rate can be expressed by the differential equation

$$\frac{dT}{dt} = k(T - A)$$

where t is time in minutes and k is constant.

(i) Show that $T = A + Ce^{kt}$ where C is a constant, is a solution of the differential equation. 1

(ii) A glass of milk warms from $4°C$ to $8°C$ in 15 minutes. The air temperature is $25°C$. Find the temperature of the glass of milk after a further 45 minutes, correct to the nearest degree. 3

(iii) With reference to the equation for T, explain the behaviour of T as t becomes very large. 1

Question 13 (15 marks) Use a SEPARATE writing booklet **Marks**

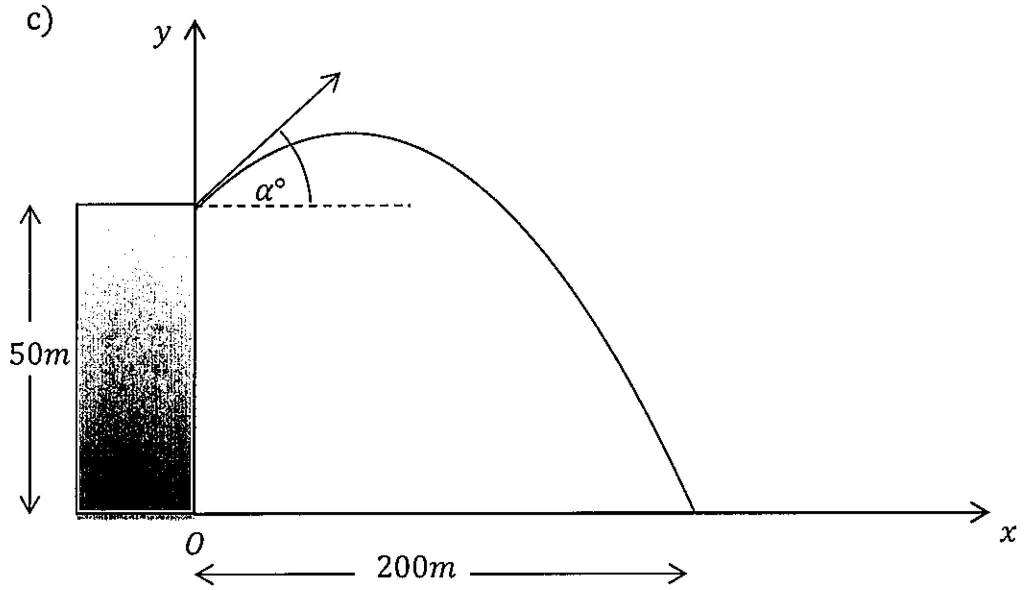
a) Evaluate $\int_0^1 x^3\left(\sqrt{x^4+1}\right)\,dx$ using the substitution $u = x^4 + 1$. 3

b) (i) By considering its second derivative, show that $y = e^x - 4x$ is always concave up. 2

(ii) Use the trapezoidal rule with 3 function values to find an approximation to $\int_3^5 (e^x - 4x)\,dx$, correct to 4 significant figures. 3

(iii) Is this approximation too large or too small? Justify your answer? 1

c)



A projectile is launched from the top of a 50 m high building with an initial speed of 40 m/s. It is launched at an angle of $\alpha°$ above the horizontal, as in the diagram. Acceleration due to gravity is 10 m/s^2.

(i) Given that $\frac{d^2x}{dt^2} = 0$ and $\frac{d^2y}{dt^2} = -10$, show that $x = 40t\cos\alpha$ and $y = -5t^2 + 40t\sin\alpha + 50$ where x and y are the horizontal and vertical displacements of the projectile in metres from 0 at time t seconds after launching. 3

(ii) The projectile lands on the ground 200 metres from the base of the building. Find the two possible values of α. Give your answers to the nearest degree. 3

Question 14 (15 marks) Use a SEPARATE writing booklet **Marks**

a) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are points on the parabola $x^2 = 4ay$.
If $p + q = 4$, find the locus of M, the mid point of PQ. 3

b) Given that x is a positive integer, prove by the method of mathematical induction that $(1+x)^n - 1$ is divisible by x for all positive integers $n \geq 1$. 3

c) The velocity v ms^{-1} of a particle moving on a horizontal line is given by

$$v^2 = 252 + 216x - 36x^2$$

(i) Show that the particle is performing simple harmonic motion. 1

(ii) Find the centre of the motion. 1

(ii) Find the amplitude of the motion. 1

(iv) Find the period of the motion. 1

(v) Find the maximum speed of the particle. 1

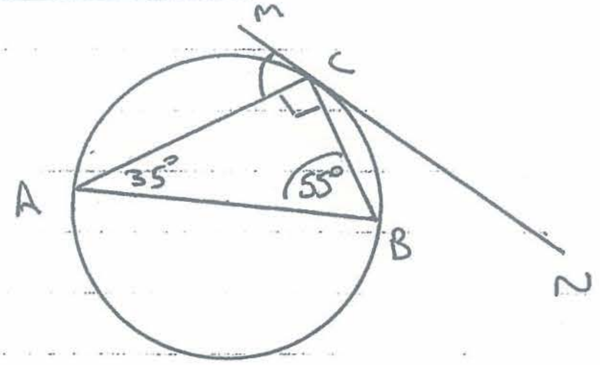
(vi) Initially the particle is at one of the extreme points of the motion. 2
Where will it be when $t = \frac{13\pi}{12}$ seconds.

(vii) Find its average speed during the first $\frac{13\pi}{12}$ seconds. 2

End of Examination

SECTION 1

1)



$\angle MCA = 55°$

$\underline{\underline{C}}$

2) $y = \sin^{-1} x$

$-1 \le x \le 1$

$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

For given graph.

$-3 \le x \le 3$

$-2\pi \le y \le +2\pi$

So $-1 \le \frac{x}{3} \le 1$

$-\frac{\pi}{2} \le \frac{y}{4} \le \frac{\pi}{2}$

$\frac{y}{4} = \sin^{-1} \frac{x}{3}$

$\therefore \underline{\underline{D}}$

3) If $y = \frac{3x-3}{x-2}$

inverse function is $x = \frac{3y-3}{y-2}$

$xy - 2x = 3y - 3$

$y(x-3) = 2x - 3$

$y = \frac{2x-3}{x-3}$

So for $f(x) = \frac{3x-3}{x-2}$

$f^{-1}(x) = \frac{2x-3}{x-3}$ $\therefore \underline{\underline{B}}$

4) $y = -x(2-x)^3(x+1)^2$

Single zero at $x=0$ ALL

Double zero at $x=-1$ A,

Triple zero at $x=2$ A,

$\therefore$ A

5) $P(2) = 0$

$$2^3 - 3(2)^2 + k(2) + 12 = 0$$
$$8 - 12 + 2k + 12 = 0$$
$$2k = -8$$
$$k = -4$$

A

6) $\tan\theta = \dfrac{m_2 - m_1}{1 + m_1 m_2}$ $\quad m_1 = 2$, $m_2 = 3$

$$= \frac{3-2}{1+3\times 2}$$
$$= \frac{1}{7}$$

$\therefore$ B

7) $2\int \cos^2 x \, dx$

$$= 2\int \left[\tfrac{1}{2}(1+\cos 2x)\right] dx$$
$$= \int (1+\cos 2x)\, dx$$
$$= x + \tfrac{1}{2}\sin 2x + c$$
$$= x + \sin x \cos x + c$$

A

8) $\dot{x} = 2e^{-x/2}$

$$\ddot{x} = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$$
$$= \frac{d\left(\frac{1}{2}\cdot 4e^{-x}\right)}{dx}$$
$$= -2e^{-x}$$

at $x = -2$ $\quad \ddot{x} = -2e^2 \quad \therefore$ C

9) $$\sin 15° = \sin(45° - 30°)$$
$$= \sin 45° \cos 30° - \sin 30° \cos 45°$$
$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{2} \times \frac{1}{\sqrt{2}}$$
$$= \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{\sqrt{6}-\sqrt{2}}{4} \quad \therefore \text{D}$$

10) $x_0 = a$ as tangent nowhere near $x = c$

A

Question 11

X1 trial 2015 Question 11 (15 marks) — Comments

Method 1

a) $\frac{1}{|x-1|} > \frac{1}{2}$ — 2 marks

NB $x \neq 1$. — $\frac{1}{2}$ mk.

Method 1: $x \neq 1$

Since $|x-1|$ is positive

$2 > |x-1|$

$|x-1| < 2$

$-2 < x-1 < 2 \quad x \neq 1$

$-1 < x < 3 \quad x \neq 1$

OR

Method 1: Students using this method were generally successful, although some lost marks for not stating $x \neq 1$.

Method 2

$x \neq 1$

if $x-1$ is negative, $x < 1$

$\frac{1}{|x-1|} > \frac{1}{2}$

$\frac{1}{-(x-1)} > \frac{1}{2}$

$\frac{1}{1-x} > \frac{1}{2}$

$2 > 1-x \quad (1-x > 0)$

$x > -1$

$\therefore x > -1$ and $x < 1$

$\therefore -1 < x < 1$

if $x-1$ is positive, $x > 1$

$\frac{1}{|x-1|} > \frac{1}{2}$

$\frac{1}{x-1} > \frac{1}{2}$

$2 > x-1$

$3 > x$

$x < 3$

$\therefore x > 1$ and $x < 3$

$\therefore 1 < x < 3$

Combining get

$-1 < x < 1$ or $1 < x < 3$

OR

$-1 < x < 3 \quad x \neq 1.$

Method 2 (Cases)

Students tried to multiply by $(x-1)^2$ on both sides as we generally do with inequalities of the form $\frac{1}{x-1} > \frac{1}{2}$

Many found it difficult dealing with $\frac{(x-1)^2}{|x-1|}$ and looking at cases although some did this successfully.

Method 3 $|x-1| < 2 \quad x \neq 1$ taking reciprocals

$-2 < x-1 < 2$

$-1 < x < 3 \quad x \neq 1.$

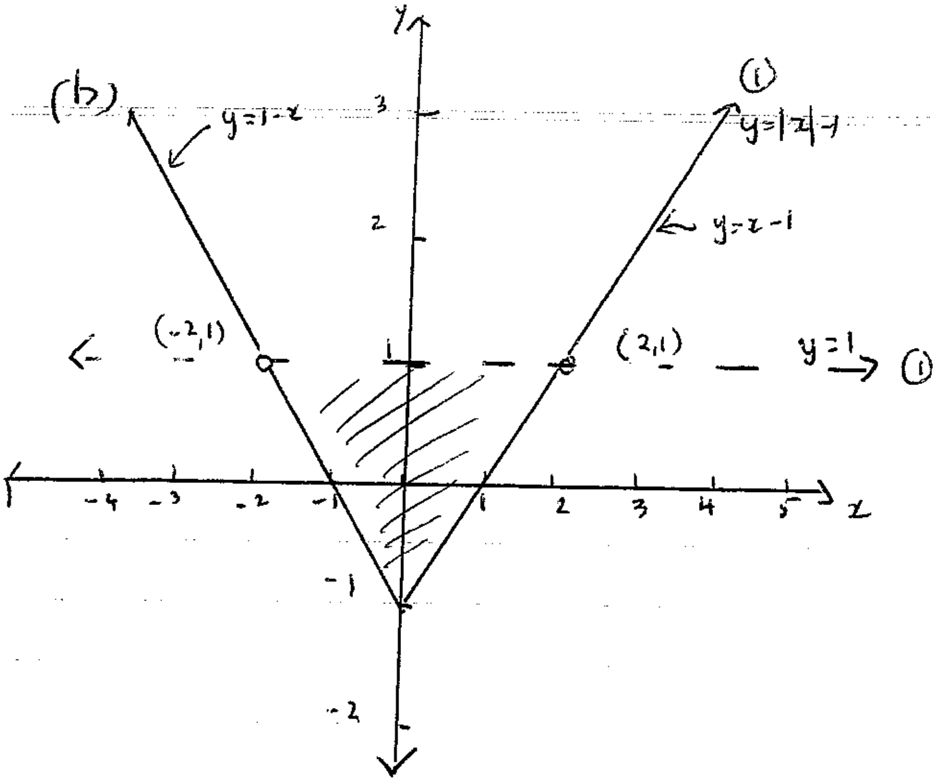
Method 3

If taking reciprocals, If $a > b > 0$ then $\frac{1}{a} < \frac{1}{b}$

NB inequalities are reversed.

Method 4: Graphical method - mainly extension 2 students.

(b)



3 marks

generally well done

- many students did NOT clearly mark the points $(-2,1)$ and $(2,1)$ with open circles and lost marks. $(\frac{1}{2})$

(c) Divide AB in the ratio 3:1

$A(-2,3)$ $\quad$ $B(10,11)$

$3 : 1$

$$\text{Point} = \left(\frac{3(10)+1(-2)}{3+1}, \frac{3(11)+1(3)}{3+1}\right) \quad ①\text{mk}$$

$$= \left(\frac{30-2}{4}, \frac{33+3}{4}\right)$$

$$= (7, 9) \quad ①\text{mk.}$$

2 marks.

- generally well done
- students who lost marks didn't correctly state or apply the formula

eg x_2 and x_1 were reversed

$k:l$

$$\left(\frac{kx_2+lx_1}{k+l}, \frac{ky_2+ly_1}{k+l}\right)$$

(d) $x_0 = 3.6$

$f(x) = x^3 - 50$

$f(3.6) = 3.6^3 - 50$

$f'(x) = 3x^2$

$f'(3.6) = 3(3.6)^2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 3.6 - \frac{3.6^3 - 50}{3.(3.6)^2} \quad ①$$

$$= 3.69 \ (2\ d/p) \quad ①$$

2 marks.

- generally well done
- some forgot the formula - many reversed the numerator & denominator in the formula.

(e) $y = \sin \ln(x)$

(i) $\frac{dy}{dx} = \cos(\ln x) \cdot \frac{1}{x}$ ① mark Well done

$= \frac{\cos(\ln x)}{x}$

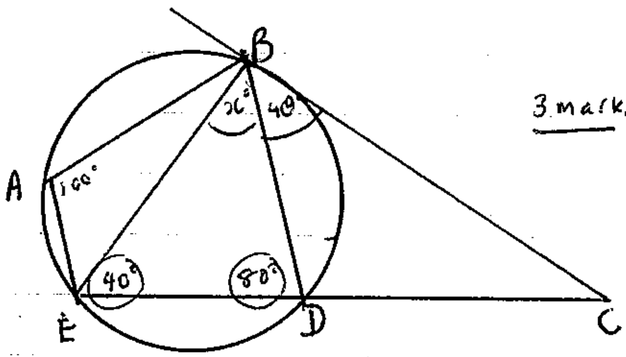
(ii) $\frac{d^2y}{dx^2} = \frac{vu' - uv'}{v^2}$ $u = \cos(\ln x)$, $u' = -\frac{\sin(\ln x)}{x}$, $v = x$, $v' = 1$

$= \frac{-\sin(\ln x) - \cos(\ln x)}{x^2}$ ①

$= -\frac{[\sin(\ln x) + \cos(\ln x)]}{x^2}$ ①

2 marks.

most mistakes confused u & V or simplified incorrectly.

(f) 

3 marks.

Well done
most used method 2

Method 1:

$\angle EBC = \angle EAB$ angle in the alternate segment.

$x + 40^\circ = 100^\circ$

$x = 60^\circ$

Method 2:

$\angle BED = \angle CBD$ angle in the alternate segment. ①

$\angle BED = 40^\circ$

$\angle EDB + \angle BAE = 180^\circ$ opposite angles of a cyclic quadrilateral are supplementary ①

$\angle EDB + 100 = 180^\circ$

$\angle EDB = 80^\circ$

$x + 40 + 80^\circ = 180^\circ$ angle sum of ΔBED ①

$x = 60^\circ$

Question 13

Solution	Comment

a) $\int_0^1 x^3 \sqrt{x^4+1}\, dx$

Let $u = x^4 + 1$

$du = 4x^3\, dx$

$\frac{1}{4}\, du = x^3\, dx$

when $x = 1,\ u = 2$

$x = 0,\ u = 1$

$$\int_1^0 x^3 \sqrt{x^4+1}\, dx = \int_1^2 \sqrt{u} \cdot \frac{1}{4}\, du$$

- 1 mark for correct substitution and change of limits

$$= \frac{1}{4}\int_1^2 u^{1/2}\, du$$

$$= \frac{1}{4}\left[\frac{2}{3} u^{3/2}\right]_1^2$$

- 1 mark - correct integration

$$= \frac{1}{6}\left[2^{3/2} - 1\right]$$

$$= \frac{1}{6}\left(2\sqrt{2} - 1\right)$$

$$= \frac{\sqrt{2}}{3} - \frac{1}{6}$$

- 1 mark for answer

3

Q13

b) i) $y = e^x - 4x$

$y' = e^x - 4$

$y'' = e^x$

as $e^x > 0$ for all values of x

$\therefore\ y'' > 0$ for all x

So the curve is always concave up.

ii) $h = \frac{b-a}{n}$

$= \frac{5-3}{2}$

$= 1$

3	4	5
y_0	y_1	y_2
(e^3-12)	(e^4-16)	(e^5-20)

$$\int_3^5 e^x - 4x\,dx \doteqdot \frac{1}{2}\left\{ y_0 + 2 \times y_1 + y_2 \right\}$$

$$= \frac{1}{2}\left\{ e^3 - 12 + 2(e^4 - 16) + e^5 - 20 \right\}$$

$$= \frac{1}{2}\left\{ 8.0855 + 2(38.598) + 128.413 \right\}$$

$$= 106.8 \quad (\text{to 4 sig figs})$$

Method 2

$$\int_3^5 (e^x - 4x)\,dx$$

$$\doteqdot \frac{4-3}{2}(e^4 - 16 + e^3 - 12) + \frac{5-4}{2}(e^5 - 20 + e^4 - 16)$$

$$\doteqdot \frac{1}{2}(e^4 + e^3 - 28) + \frac{1}{2}(e^5 + e^4 - 36)$$

$$\doteqdot 106.8$$

This was answered well.

- 1 mark for derivatives
- 1 mark for stating $e^x > 0$ and $y'' > 0$

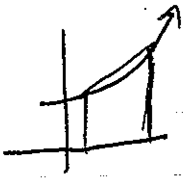
2

- 1 mark for finding h correctly
- 1 mark for correctly using trapezoidal rule
- 1 mark for answer

3

Q 13

b) iii) It is too large as the trapezium area is <u>larger</u> than the area under the curve as the function concaves up for all x and $f(3) < f(5)$ so the function has



 shape

1 mark

<u>Note</u>: For those students who only said it was too large without a reason, I awarded only ½ mark.

Examiners comment for

13 b ii) The approximation can be found two ways:

Method 1 Using the formula for the trapezoidal rule for n-equal subintervals

$$\frac{h}{2}\left\{ y_0 + 2y_1 + y_2 \right\}$$

y_0 y_1 y_2

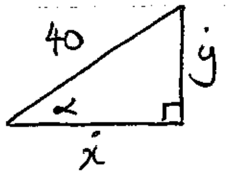
and

Method 2 : Using one application of trapezoidal rule twice.

Some Students who used Method 1, did not find h ~~exact~~ correctly and confused the rule by multiply y_2 by 2 instead of y_1.

Q13

c) i) Initial conditions, $t=0$



$v_x \quad \dot{x} = 40\cos\alpha$

$v_y \quad \dot{y} = 40\sin\alpha$

Horizontal component

$\ddot{x} = 0$

$\dot{x} = c$

when $t=0$, $\dot{x} = 40\cos\alpha$

$\therefore c = 40\cos\alpha$

$\therefore \dot{x} = 40\cos\alpha$

$x = 40t\cos\alpha + c$

when $t=0$, $x=0$,

$\therefore c = 0$

$\therefore x = 40t\cos\alpha \quad \text{--(1)}$

Vertical

$\ddot{y} = -10$

$\dot{y} = -10t + c$

when $t=0$, $\dot{y} = 40\sin\alpha$

$\therefore c = 40\sin\alpha$

$\therefore \dot{y} = -10t + 40\sin\alpha$

$y = -5t^2 + 40t\sin\alpha + C$

when $t=0$, $y=50$

$50 = C$

$\therefore y = -5t^2 + 40t\sin\alpha + 50 \quad \text{--(2)}$

- 1 for Initial Conditions
- Students were penalised for not finding c.
- 1 for $\dot{x}$ + $\dot{y}$
- 1 to find c for x and y.

3

ii) When $x = 200$, $y = 0$

Using $x = 40t\cos\alpha$

$200 = 40t\cos\alpha$

$t = \dfrac{5}{\cos\alpha}$

sub in t and $y=0$ in (2)

$0 = -5\left(\dfrac{5}{\cos\alpha}\right)^2 + 40\left(\dfrac{5}{\cos\alpha}\right)\sin\alpha + 50$

$= -125\sec^2\alpha + 200\tan\alpha + 50$

$= -125(1+\tan^2\alpha) + 200\tan\alpha + 50$

$= -125 - 125\tan^2\alpha + 200\tan\alpha + 50$

$= 5\tan^2\alpha - 8\tan\alpha + 3$

$= (5\tan\alpha - 3)(\tan\alpha - 1)$

$\tan\alpha = \dfrac{3}{5}$ or $\tan\alpha = 1$

$\alpha = 31^\circ$ (nearest degree) $\qquad \alpha = 45^\circ$

1 sub $x=200$ in (1)

1 sub. $y=0$ and t in (2)

1 for correct angles

3

Alternative solution to c) ii)

From (1)

$$t = \frac{x}{40\cos\alpha}$$

Sub in (2)

$$y = -5\left(\frac{x}{40\cos\alpha}\right)^2 + 40\left(\frac{x}{40\cos\alpha}\right)\sin\alpha + 50$$

$$= \frac{-5x^2}{1600\cos^2\alpha} + x\tan\alpha + 50$$

$$= \frac{-x^2\sec^2\alpha}{320} + x\tan\alpha + 50$$

$$= \frac{-x^2(1+\tan^2\alpha)}{320} + x\tan\alpha + 50$$

when $y = 0$

$$0 = -x^2 - x^2\tan^2\alpha + 320x\tan\alpha + 16000$$

$$0 = x^2 + x^2\tan^2\alpha - 320x\tan\alpha - 16000$$

$$0 = x^2\tan^2\alpha - 320x\tan\alpha + x^2 - 16000$$

but $x = 200$

$$0 = 40\,000\tan^2\alpha - 64000\tan\alpha + 24000$$

$$= 5\tan^2\alpha - 8\tan\alpha - 3$$

Then as per method 1

This method was not the optimal method. It allowed for more errors

1 mark was awarded here, when $t = x/40\cos\alpha$ was substituded in (1)

1 mark was was awarded when $y=0$, $x=200$ was subst. (No marks were awarded when students took $y = -50$).

1 mark for correct angles

QUESTION 12

SOLUTION	COMMENTS

SOLUTION

(a) Let $R\sin(x+\alpha) = 3\sin x + 4\cos x$ — ①

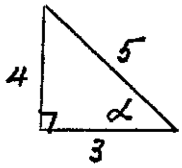
$\therefore R\sin x\cos\alpha + R\cos x\sin\alpha$

$= 3\sin x + 4\cos x.$

$$\left.\begin{aligned}\therefore R\cos\alpha &= 3\\ R\sin\alpha &= 4\end{aligned}\right\} \Rightarrow \alpha \text{ is acute}$$

$$\therefore \tan\alpha = \frac{4}{3}$$

$$\alpha = \tan^{-1}\left(\frac{4}{3}\right)$$



$$\therefore R \times \frac{3}{5} = 3$$

$$R = 5$$

① $\Rightarrow$ $5\sin(x+\alpha) = 2.5$

$$\sin(x+\alpha) = 0.5$$

$$\therefore x+\alpha = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \ldots$$

$$\therefore x = \frac{\pi}{6} - \tan^{-1}\left(\frac{4}{3}\right), \frac{5\pi}{6} - \tan^{-1}\left(\frac{4}{3}\right)$$

$$\frac{13\pi}{6} - \tan^{-1}\frac{4}{3}, \ldots$$

$$= -0.41, 1.69, 5.88, \ldots$$

(correct to 2 dec. places)

BUT $0 \le x \le 2\pi \Rightarrow x = 1.69, 5.88.$

OR Using t-substitution

$$3\sin x + 4\cos x = 2.5$$

$$\Rightarrow 3.\frac{2t}{1+t^2} + 4.\frac{1-t^2}{1+t^2} = 2.5$$

$$\Rightarrow 6t + 4 - 4t^2 = 2.5 + 2.5t^2$$

$$0 = 6.5t^2 - 6t - 1.5$$

ie $13t^2 - 12t - 3 = 0$

$$\therefore t = \frac{12 \pm \sqrt{144+156}}{26}$$

$$= \frac{12 \pm 10\sqrt{3}}{26}$$

$$= \frac{6 \pm 5\sqrt{3}}{13}$$

COMMENTS

• Some students squared both sides but did not check their solutions.

• 1 MARK for $5\sin(x + \tan^{-1}\frac{4}{3}) = 2.5$

• Some students made no mention of radians.

• Many students missed the solution $x = \frac{13\pi}{6} - \tan^{-1}\left(\frac{4}{3}\right)$ and did not realise that $x = \frac{\pi}{6} - \tan^{-1}\left(\frac{4}{3}\right)$ was outside the domain. (2 MARKS)

SOLUTION	COMMENTS
$\therefore \tan\frac{x}{2} = 1.1277, -0.2046$ $\therefore x = 1.69, 5.88$ (correct to 2 dec. pl)	• need to check $x=\pi$ when using this method.
(b) (i) (ii) $AP^2 = PC.PD$ $= 12\times 5$ $= 60$ $AP > 0 \Rightarrow AP = \sqrt{60}$ $= 2\sqrt{15}$	• All students were successful! • Too many students wrote $AP^2 = 7\times 5$.
(iii) $A\hat{B}D = B\hat{A}Q = \alpha$ (alternate angles, $DB \parallel AQ$) $A\hat{D}B = B\hat{A}Q = \alpha$ (angle between tangent and chord equals angle in alternate segment) $\therefore B\hat{A}D = 180° - 2\alpha$ (angle sum of triangle ABD) $B\hat{A}D + B\hat{C}D = 180°$ (opposite ∠'s of cyclic quad are supplementary) $\therefore 180 - 2\alpha + B\hat{C}D = 180°$ $\therefore B\hat{C}D = 2\alpha$	• Generally gained at least 1 mark for correct statement and reason.
(iv) $QA = QB$ (tangents to circle from external point are equal) $\therefore Q\hat{B}A = Q\hat{A}B = \alpha$ (equal angles opposite equal sides in isosceles triangle) $\therefore A\hat{Q}B = 180° - 2\alpha$ (angle sum of triangle is 180°) $A\hat{Q}B + B\hat{C}D = 180° - 2\alpha + 2\alpha$ $= 180°$ $\therefore PQBC$ is cyclic quad. as opposite angles are supplementary	• Generally lose 1 mark for any missing reason.

	COMMENTS.
(c) $\dfrac{dT}{dt} = k(T-A)$ (i) If $T = A + Ce^{kt}$ —— (1) then $\dfrac{dT}{dt} = Cke^{kt}$ $= k.Ce^{kt}$ $= k(T-A)$ from (1)	• Generally well done – no need to DERIVE the result by integration.
(ii) $T = A + Ce^{kt}$ $A = 25$ at $t=0, T=4$ $\therefore 4 = 25 + Ce^{0}$ $\therefore C = -21$ $\therefore T = 25 - 21e^{kt}$ $\left.\begin{matrix} t=15 \\ T=8 \end{matrix}\right\} \Rightarrow 8 = 25 - 21e^{15k}$ $\Rightarrow e^{15k} = \dfrac{17}{21}$ $\therefore k = \dfrac{\ln\left(\frac{17}{21}\right)}{15}$ —— (2) at $t=60$ $T = 25 - 21e^{60k}$ $= 15.98\,^{\circ}C$ (2 dec. pl) $= 16^{\circ}C$ to nearest degree	• Some students were unable to begin this question, others were confused, thought that $t = 15$ when $T = 4$ • Find C correctly 1 MARK • Find k correctly 1 MARK • Use of $t = 45$ scored no mark.
(iii) $T = 25 - 21e^{kt}$ From (2) $k < 0$ Hence as $t \rightarrow$ $T \rightarrow 25$ as $e^{kt} \rightarrow 0$	• Some students thought that $T \rightarrow \infty$ as $t \rightarrow \infty$ because they failed to see that $k < 0$.

QUESTION 14

a) $M = \left(\frac{2ap+2aq}{2}, \frac{ap^2+aq^2}{2} \right)$ 1 mark

$\therefore x = \frac{2ap+2aq}{2}$

$= a(p+q)$

$= 4a$

1mark for correctly using $p+q=4$

$\therefore$ The locus of M is the vertical line $x=4a$

1mark for concluding that the locus is $x=4a$. Note that this had to be your answer, not just floating around in your working.

b) Prove true for $n=1$

when $n=1$,

$(1+x)^1 - 1 = 1 + x - 1$

$= x$

which is divisible by x

Assume true for $n=k$

That is,

$(1+x)^K - 1 = Mx$,

where M is an integer

1mark for correctly proving true for $n=1$ and making the assumption

Prove true for $n=k+1$

That is,

$(1+x)^{k+1}-1$ is divisible by x

$$(1+x)^{k+1}-1 = (1+x)^k(1+x)-1$$
$$= (1+x)^k + x(1+x)^k - 1$$
$$= x(1+x)^k + (1+x)^k - 1$$
$$= x(1+x)^k + Mx$$

by assumption

$$= x\left((1+x)^k + M\right)$$

1 mark for correctly using your assumption.

which is divisible by x.

$\therefore$ the statement is true for $n=k+1$ if it is true for $n=k$.

It is true for 1, so it is true for 2, and therefore true for all n.

1 mark for reducing to something obviously divisible by x, and for concluding with an appropriate statement.

c)

i) $v^2 = 252 + 216x - 36x^2$

$\frac{1}{2}v^2 = 126 + 108x - 18x^2$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(126 + 108x - 18x^2\right)$

$= 108 - 36x$

$= -36(x-3)$ —— 1

which is of the form

$\ddot{x} = -n^2(x - x_0)$

$\therefore$ the particle is in simple harmonic motion

OR

$v^2 = 252 + 216x - 36x^2$

$= 36(7 + 6x - x^2)$

$= 36(7 + 9 - 9 + 6x - x^2)$

$= 36[16 - (x^2 - 6x + 9)]$

$= 36[16 - (x-3)^2]$ —— 1

which is of the form

$n^2(a^2 - (x - x_0)^2)$

$\therefore$ simple harmonic motion

This is a "show" question, and there is very little flexibility in the form your answer must take

[$36(3-x)$ is worth half marks]

ii) $x = 3\,\text{m}$ 1 mark

iii) Extremes when $v = 0$

$$36x^2 - 216x - 252 = 0$$
$$x^2 - 6x - 7 = 0$$
$$(x-7)(x+1) = 0$$
$$\therefore x = 7 \text{ or } x = -1$$

Amplitude $= \frac{7 - -1}{2}$

$= 4\text{m}$ ~~units~~ 1 mark

iv) $n^2 = 36$ so $n = 6$

Period $= \frac{2\pi}{n}$

$= \frac{2\pi}{6}$

$= \frac{\pi}{3}$ s 1 mark

v) Maximum speed at centre
ie when $x = 3$

$v^2 = 252 + 648 - 324$

$= 576$

$v = \pm 24$ but speed > 0

$\therefore$ maximum speed is $24\,\text{ms}^{-1}$

The question asks for speed, so don't give $|V_{max}|$ or similar.

1 mark

vi) Motion of the form

$$x = x_0 + a\cos(nt)$$
$$= 3 + 4\cos 6t$$

1 mark

when $t = \frac{\cdot\cdot}{12}$,

$$x = 3 + 4\cos\left(6 \times \frac{13\pi}{12}\right)$$

$$= 3 + 4\cos\frac{13\pi}{2}$$

$$= 3 + 4 \times 0$$

$$= 3\text{ m}$$

1 mark

vii) As period is $\frac{\pi}{3}$,

$$\frac{13\pi}{12} \div \frac{\pi}{3} = \frac{13}{4}$$

$$= 3\tfrac{1}{4}$$

$\therefore$ The particle has completed $3\frac{1}{4}$ oscillations

1 oscillation is $4 \times$ amplitude $= 16\text{m}$

$\therefore$ total distance $= 3\frac{1}{4} \times 16$

$= 52\text{m}$

1 mark

This question requires some understanding of periodicity – finding average speed without that is worth zero

$$\text{Average speed} = \frac{\text{distance}}{\text{time}}$$

$$= \frac{52}{13\pi/12}$$

$$= \frac{48}{\pi}\text{ ms}^{-1}$$

1 mark